# SECONDARY WAVES IN TWO-COMPONENT MEDIA

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In this paper we calculate the potentials of the secondary waves which appear in the reflection of a spherical sound wave from the plane dividing surface between a fluid and a two-component medium, composed of a fluid and an elastic component (moist earth, porous sound-absorbing materials, pulp, etc.). The dimensions of pores and solid particles are assumed to be small compared to the distances over which the kinematic and dynamic characteristics of the motion change significantly, so that both components of the medium may be considered to be continua. The dynamics of such a medium have been considered in a number of papers [1-4]. In [5] it was shown that the equations [2] are the most general for the case of harmonic waves. These equations are used as the point of departure in the present note. The two-component medium is taken to be homogeneous and isotropic.

Let there be a point source of sound at the point $O$ (Fig. 1) in the fluid, distance $y$ from the dividing surface. The potential of the reflected wave has the form [6]

$$\varphi = \frac{ik_0}{2} \int\limits_{-\pi/2+i\infty}^{\pi/2-i\infty} H_0^{(1)}(u) \exp\left[ik_0 (y + y_0)\cos\theta\right] W(\theta) \sin\theta \, d\theta \quad (u = k_0 r \sin\theta) \qquad (1)$$

Here $k_0$ is the modulus of the wave vector in the fluid, $H_0^{(1)}$ is the Hankel function of first kind, of zero order, $W(\theta)$ is the coefficient of reflection of a plane wave impinging on the boundary at the angle $\theta$. For $\theta = 0$ this coefficient is equal to [7]

$$W = \frac{Z - Z_0}{Z + Z_0}, \qquad \frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \qquad (2)$$

Here $Z_0 = \rho_0 c_0$ is the impedance of the "upper" fluid medium ($\rho_0$ and $c_0$ are, respectively, the density and sound speed in the fluid); $Z_1 = \rho_1 c_1$ and $Z_2 = \rho_2 c_2$ are the "effective" impedances of the elastic and fluid components of the "lower" medium; $c_1$ and $c_2$ are the speeds of longitudinal waves of first and

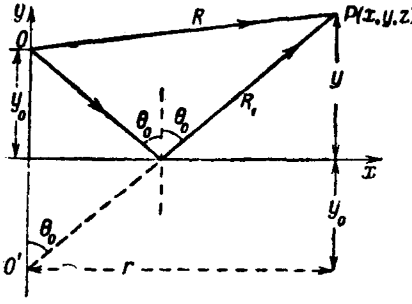


Fig. 1.

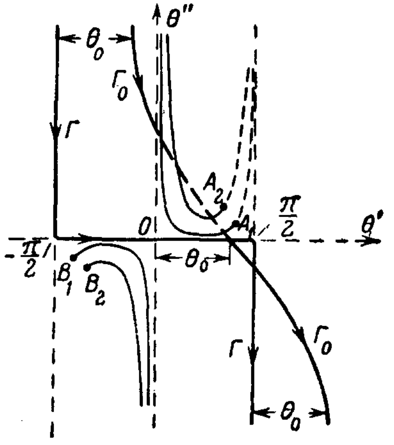


Fig. 2.

second type, while the coefficients $\rho_1$ and $\rho_2$, which depend on the porosities, densities, and elastic parameters of both components of the "lower" medium, may be treated as some "effective" densities of those components. If shear waves are neglected, equations (2) are still valid for incidence at any angle, but now

$$Z_0 = \rho_0 c_0 \sec\theta, \qquad Z_1 = \rho_1 c_1 \sec\theta_1, \qquad Z_2 = \rho_2 c_2 \sec\theta_2 \tag{3}$$

Making use of the relations

$$\frac{\sin\theta}{c_0} = \frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

we write equations (2) for $W(\theta)$ in the form

$$W(\theta) = \frac{\cos\theta - m_1\sqrt{n_1^2 - \sin^2\theta} - m_2\sqrt{n_2^2 - \sin^2\theta}}{\cos\theta + m_1\sqrt{n_1^2 - \sin^2\theta} + m_2\sqrt{n_2^2 - \sin^2\theta}} \tag{4}$$

Here

$$n_1 = c_0 / c_1, \qquad n_2 = c_0 / c_2, \qquad m_1 = \rho_0 / \rho_1, \qquad m_2 = \rho_0 / \rho_2$$

To obtain the potential of the reflected spherical wave at distances large compared to the wave length, an asymptotic form of the Hankel function is used

$$H_0^{(1)}(u) \approx \sqrt{\frac{2}{\pi u}} \exp\left[i\left(u - \frac{\pi}{4}\right)\right]\left(1 + \frac{1}{8iu}\right) \tag{5}$$

Putting (5) in (1), and noting that

$$y + y_0 = R_1 \cos\theta_0, \qquad r = R_1 \sin\theta_0$$

we obtain

$$\varphi = \left(\frac{k_0}{2\pi r}\right)^{1/2} \exp\frac{i\pi}{4} \int_\Gamma \exp\left[ik_0 R_1 \cos(\theta - \theta_0)\right]\left(1 + \frac{1}{8ik_0 r \sin\theta}\right) W(\theta) \sqrt{\sin\theta}\, d\theta \quad (6)$$

where $\Gamma$ denotes the integration path, going from the point $-1/2\,\pi + i\infty$ to the point $\pi/2 - i\infty$ (Fig. 2). The integral in (6) is easily evaluated by the method of steepest descent [6]. The path of steepest descent $\Gamma_0$ goes through the characteristic point $\theta_0$ and departs from it along the line

$$\operatorname{Im} i \cos(\theta - \theta_0) = \cos(\theta' - \theta_0)\cosh\theta'' = 1, \qquad \theta = \theta' + i\theta''$$

This path intersects the real axis at the point $\theta_0$ at an angle of $45^\circ$ and goes, on one side, to $-1/2\,\pi + \theta_0 + i\infty$, and, on the other $1/2\,\pi + \theta_0 - i\infty$. The function $W(\theta)$ has the roots $\sqrt{(n_1^2 - \sin^2\theta)}$ and $\sqrt{(n_2^2 - \sin^2\theta)}$, and thus the points $\theta = \pm\sin^{-1} n_1$ and $\theta = \pm \sin^{-1} n_2$ will be branch points.

We make cuts in the complex plane along the lines

$$n_1^2 - \sin^2\theta = x_1^2, \qquad n_2^2 - \sin^2\theta = x_2^2$$

Here $x_1^2$ and $x_2^2$ are real positive quantities, within the limits $(0, \infty)$. The values $x_1 = 0$ correspond to the branch points. For $x_1^2 \to \infty$, $x_2^2 \to \infty$, we have, for both cuts, $\sin\theta \to \pm i\infty$; from here it follows that $\theta' \to 0$, $\theta'' \to \pm\infty$. These cuts are shown in Fig. 2 ($A_1B_1$ and $A_2B_2$, corresponding to the branch points of the first and second roots). If the points $A_1$ and $A_2$ are located between $\Gamma$ and $\Gamma_0$, the integral along $\Gamma$ will be equal to the integral along $\Gamma_0$ plus integrals along the edges of the cuts. As a result, the full expression for the reflected wave will be composed of three parts

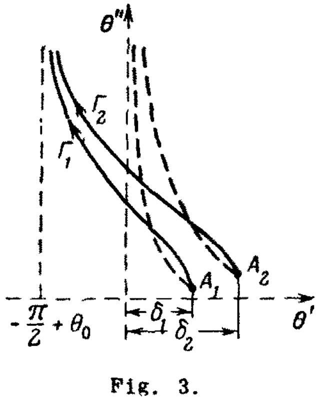

Fig. 3.

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2$$

where $\varphi_0$ is the reflected wave proper and $\varphi_1$ and $\varphi_2$ are the first and second secondary waves.

For example, in the case where the imaginary parts of $n_1$ and $n_2$ are vanishingly small and the real parts are less than unity, we shall have, for this location of $A_1$ and $A_2$

$$\theta_0 > \delta_1 = \sin^{-1} n_1, \qquad \theta_0 > \delta_2 = \sin^{-1} n_2$$

Thus $\theta_0$ must be greater than the angles of total internal reflection of longitudinal waves of first and second types.

The function $W(\theta)$ also has singular points in the form of poles, which are found from the equation

$$\cos\theta + m_1\sqrt{n_1^2 - \sin^2\theta} + m_2\sqrt{n_2^2 - \sin^2\theta} = 0$$

If these points lie between $\Gamma$ and $\Gamma_0$, the expression for the potential of the reflected wave will contain, besides the above-mentioned parts $\varphi_0$, $\varphi_1$ and $\varphi_2$, terms coming from an evaluation of the integrand at the poles. This question is not investigated here.

Evaluation of the integral along $\Gamma_0$ gives [6]

$$\varphi_0 = \frac{\exp(ik_0R_1)}{R_1}\left[W(\theta_0) - \frac{iN}{k_0R_1}\right], \qquad N = \frac{1}{2}\left[W''(\theta_0) + W'(\theta_0)\cot\theta_0\right]$$

Here $W'(\theta_0)$ and $W''(\theta_0)$ are derivatives of the coefficient of reflection with respect to the angle $\theta$, taken at the point $\theta = \theta_0$.

Neglecting the quantity $1/8\ k_0r\sin\theta$ in (6) in comparison with unity, we obtain

$$\varphi_1 = \left(\frac{k_0}{2\pi r}\right)^{1/2}\exp\frac{i\pi}{4}\left(\int_{i\infty}^{A_1}\exp\left[ik_0R_1\cos(\theta-\theta_0)\right]W(\theta)\sqrt{\sin\theta}\,d\theta + \right.$$

$$\left. + \int_{A_1}^{i\infty}\exp\left[ik_0R_1\cos(\theta-\theta_0)\right]W^+(\theta)\sqrt{\sin\theta}\,d\theta\right)$$

Here $W(\theta)$ is the value of the coefficient of reflection on the left edge of the cut $W^+(\theta)$ on the right edge. These values differ in the sign of the root, $\sqrt{(n_1^2 - \sin^2\theta)}$. Interchanging the limits of integration in the first integral, the two integrals are reduced to one

$$\varphi_1 = \left(\frac{k_0}{2\pi r}\right)^{1/2}\exp\frac{i\pi}{4}\int_{A_1}^{i\infty}\exp\left[ik_0R_1\cos(\theta-\theta_0)\right]\Phi_1(\theta)\sqrt{\sin\theta}\,d\theta \qquad (7)$$

$$\Phi_1(\theta) = W^+(\theta) - W(\theta) = \frac{4m_1\cos\theta\sqrt{n_1^2 - \sin^2\theta}}{(\cos\theta + m_2\sqrt{n_2^2 - \sin^2\theta})^2 - m_1^2(n_1^2 - \sin^2\theta)}$$

The integral in (7) is evaluated by the method of steepest descent [6]; we deform the path of integration in such a way that, from the point $A_1$, it goes along the line on which the exponent under the integral falls the fastest. This will be the line where

$$\operatorname{Re}\cos(\theta - \theta_0) = \text{const} \tag{8}$$

Here, it is necessary that

$$\operatorname{Im}\cos(\theta - \theta_0) > 0 \tag{9}$$

We shall assume that $n_1$ is a real quantity. Since at point $A_1$ the angle $\theta = \delta_1 = \sin^{-1} n_1$, equation (8) takes the form

$$\operatorname{Re}\cos(\theta - \theta_0) = \cos(\delta_1 - \theta_0) \quad \text{or} \quad \cos(\theta_0 - \theta')\cosh\theta'' = \cos(\delta_1 - \theta_0)$$

On this path of integration (we denote it by $\Gamma_1$), condition (9) is also satisfied. Thus, we have

$$\varphi_1 = \left(\frac{k_0}{2\pi r}\right)^{1/2} \exp\left[ik_0R_1\cos(\delta_1 - \theta_0) + \frac{i\pi}{4}\right] \times$$
$$\times \int_{\Gamma_1} \Phi_1(\theta) \exp\left[-k_0R_1\sin(\theta_0 - \theta')\sinh\theta''\right] \sqrt{\sin\theta}\, d\theta$$

Since $\Gamma_1$ is the path of steepest descent, the value of the integral is determined mainly by the initial portion of the path. Therefore, we may put $\theta' = \delta_1$ under the integral, and assume $\theta''$ to be small. Noting that then

$$\Phi_1(\theta) = -\frac{4m_1\cos\delta_1\sqrt{-2in_1\cos\delta_1}}{(\cos\delta_1 + m_2\sqrt{n_2^2 - n_1^2})^2}\sqrt{\theta''}$$

we obtain

$$\varphi_1 = -4m_1\left(\frac{k_0}{\pi r}\right)^{1/2} \exp\left[ik_0R_1\cos(\delta_1 - \theta_0)\right] \frac{n_1}{\sqrt{\cos\delta_1}\,\left[1 + m_2\sqrt{(n_2^2 - n_1^2)/(1 - n_1^2)}\right]^2} \times$$
$$\times \int_0^\infty \exp\left[-\theta'' k_0R_1\sin(\theta_0 - \delta_1)\right] \sqrt{\theta''}\, i d\theta''$$

or, finally

$$\varphi_1 = -\frac{2im_1n_1\exp\left[ik_0R_1\cos(\delta_1 - \theta_0)\right]}{k_0\sqrt{r\cos\delta_1}\,\left[1 + m_2\sqrt{(n_2^2 - n_1^2)/(1 - n_1^2)}\right]^2\left[R_1\sin(\theta_0 - \delta_1)\right]^{3/2}} \tag{10}$$

In a similar way, the potential for the second secondary wave is found to be

$$\varphi_2 = -\frac{2im_2n_2\exp\left[ik_0R_1\cos(\delta_2 - \theta_0)\right]}{k_0\sqrt{r\cos\delta_2}\,\left[1 + m_1\sqrt{(n_1^2 - n_2^2)/(1 - n_2^2)}\right]^2\left[R_1\sin(\theta_0 - \delta_2)\right]^{3/2}} \tag{11}$$

In the limiting cases of vanishing porosity, and of porosity approaching unity, we will have, respectively, [7]

$$m_1 = \rho_0/\rho_s, \quad m_2 = 0, \qquad m_1 = 0, \quad m_2 = \rho_0/\rho_f$$

where $\rho_s$ and $\rho_f$ are the actual densities of the elastic and fluid components of the medium. Then, equations (10) and (11) reduce to the well-known expression for the potential of the secondary wave which appears at the dividing boundary between two continuous media.

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